LEARNING ABOUT PRIMES

Alec McEachran describes a teaching approach to prime numbers which does more than just define and list them.

I first learned that there were numbers called 'prime' at secondary school. Unfortunately, I am not sure that I learned *about* prime numbers in those lessons. We defined and listed them and at some later point were tested on our memory of both the definition and the list. This unhappy experience caused me to consider a different approach when I ended up teaching the same topic as a new maths teacher. The creative process of preparing this topic had unexpected results.

To introduce primes, our teacher first told us a definition of a prime number: a prime number is an integer bigger than one which can only be divided by 1 and by itself.[1] Having been told what they were, we were then taught how to find prime numbers: You find prime numbers using the Sieve of Eratosthenes (*figure 1*)![2] We had been given an algorithm to follow, and, as far as I recall, the algorithm was the lesson. Our only measures of success were how *fast*, how *accurate* and how *neatly* we could find the teacher's list of numbers. There was no meaning to the numbers on the list other than that it was produced by this particular algorithm.

Something had gone wrong. It surely cannot be that the objectives for a class on prime numbers were to follow an algorithm, practise repeated addition and create a list of numbers? What were we supposed to learn? The expectation of being tested encouraged us to learn this list of primes, but we gained no real understanding of the subject. It is easy for me to understand why students often hate this subject; I was such a student.

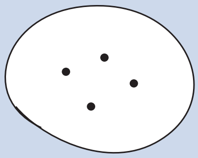
Having subsequently been 'born again' to maths, and having trained as a teacher, I find it frustrating that I wasted so much time in these classes. So, as the teacher, I wanted to turn the lesson about and see if students could themselves discover the Sieve of Eratosthenes, or at least experience something of what Eratosthenes must have experienced when he was studying this area of mathematics. So we began as shown in *figure 2*.

The hope is that this investigation leads students to perceive that numbers seem to fall into two categories. They should notice that certain numbers (1,2,3,5,7,11,. . .) cannot be described in terms of smaller groupings. Hopefully, students will also realise that (nearly) the same set of the

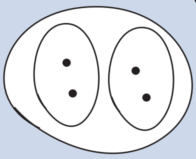
21	22	23	24	25	26	27	28	29	30
11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10

figure 1

Today we are going to try to get rid of certain numbers by describing them in a different way.



Here is a collection of 4 pebbles. I can split the pebbles into two equally-sized groups. Instead of saying *"I have four pebbles"* I can say *"I have two groups of two pebbles"*. So we replace the word *"four"* with *"two groups of two"*.



There is one main rule: you can break collections only into equally-sized groups.

Starting Points:

- How can we describe a collection of six pebbles?
- What about 10 pebbles?
- Investigate other numbers of pebbles.

Keep a record of everything you do and anything that you notice.

figure 2

numbers (2,3,5,7,11,. . .) are also those which appear in the descriptions of all the other quantities too.

At an appropriate point, the teacher can then ask:

Are all the different ways you can describe 12 pebbles equivalent?

12 is an interesting number, because it has three prime factors: $12 = 2\times2\times3$.[3] Different students are almost certain to have split 12 in different ways: $2\times3\times2$, $2\times2\times3$, $3\times2\times2$. Constructed visually, these look quite different. At this point, it might be worth pausing to note that multiplication is commutative. (There is a potentially interesting extension into permuta-

tions here.) It is also likely that some students have decided to describe 12 as '2 groups of 6'. For the investigation to lead towards prime numbers, we need those students to make a leap to describe 12 as '2 groups of 2 groups of 3' *and to understand why that description is more satisfactory*.

Making this move is the crux of the investigation. The way the initial instructions are worded is intended to move students to realise that once 'six' can be described as 'two groups of three', using 'six' elsewhere is against the spirit of the investigation. Students who find this difficult can be cajoled linguistically and visually:

'Twelve' is 'two groups of six'.
but *'Six' is 'two groups of three'.*
so *'Twelve' is 'two groups of … two groups of three'.*

It should be recognised that this language may feel alien to youngsters, because it has a nested grammatical structure that is not familiar in everyday speech.[4]

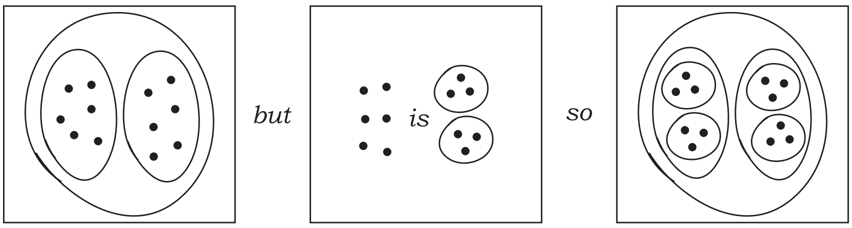


figure 3

The visual argument proceeds similarly (*figure 3*). There is a danger here of reverting to the kind of telling that I was subjected to as a learner. The last thing I want to do is to convince students that *I am right*: I want them to convince themselves that what I've said is right. Students accept the 'argument from authority' every day, but it doesn't mean that they internalise what they accept.[5]

So what are these numbers?

If and when this 'move' is accepted by students, it should strengthen the sense that numbers fall into two camps: the 'atomic' and the 'compound'. If the students are lucky, they might then have come to some sort of understanding about prime numbers without knowing what a 'prime number' is. It would be interesting for students who don't know the name 'prime' to come up with their own names for these two groups at this point.

The purpose of the investigation is for students to understand what these numbers *are*, not what they are *called*. When I was a student, I was told: "There are these things called 'primes'…" but we never really knew any of their properties. If this investigation is successful, students will have discovered a set of numbers for themselves which have particular characteristics, and only then are told: "We call these numbers *prime*". Few toddlers learn the definition of 'tree', but rather, on seeing some trees, are told: "We call those 'trees'."

How would you go about working out what groupings describe 119, 247 and 289 pebbles?

Finding factors of larger numbers like these ($119=7\times17$, $247=13\times19$, $189=17\times17$) requires a more methodical approach. It is possible that the students will understand that testing the divisibility of the number by each known prime in turn will eventually yield either a new prime or a factor. Perhaps not – but if younger students can get there (or even begin to get there) then it makes a mockery of the idea that to teach prime numbers one first tells the students about an algorithm for finding them.

Teaching variations of this idea to different KS3 classes, I have become increasingly taken with the visualisation of numbers as nested sets of units. They produce structured, beautiful visualisations. *Figure 4* shows visualisations of the numbers 28, 35 and 30. Within the outer circle of each visualisation are 28, 35 and 30 small black dots respectively,

figure 4

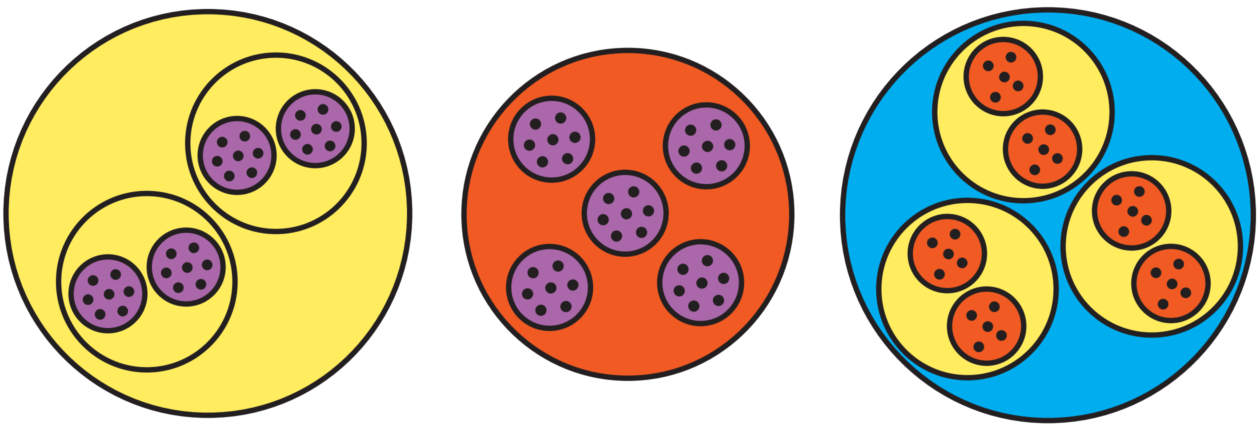
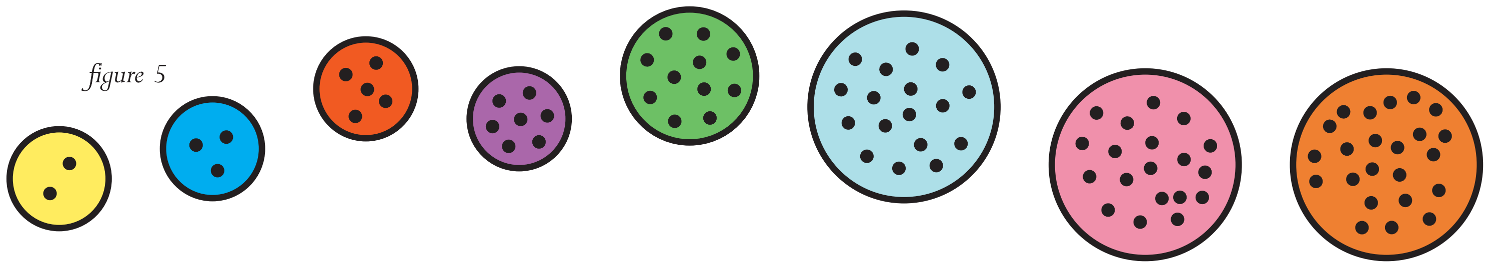


figure 5



each representing a unit. The units are grouped together in colour-coded prime-sized sets. For 28, 30 and any other number with more than two factors, these prime-sized sets are themselves collected into colour-coded prime-sized sets. The colour coding that I have adopted is that a group of two is yellow, a group of three is blue, five red, seven purple, and so on (see *figure* 5). Each new prime adds a new colour to the mix. I call these visualisations *primitives*.

Each primitive seems to contain a huge amount of data – far more than the decimal coding '30' or '35'. In the visualisation of 30, for example, I can perceive quickly many different sums and products. Six fives, three tens and thirty-units are all perceptible. It is refreshing to see 5×7 and 35 simultaneously, rather than thinking of one as an 'answer' to the other. In the same way that students beginning to learn algebra have to come to understand that the equals sign isn't really an instruction to solve but denotes an equality, the visualisations do not seek to resolve the multiplications within them, but embrace the multiplications and their resolutions together.

Particular numbers take on peculiar features when numbers are perceived in this way. Primes and powers of primes have only one visualisation, while other numbers have several. The power series 2,4,8,16,32,64,… feel austere and fractal-like. By contrast, 6, 30, and 210 (2×3, 2×3×5, 2×3×5×7) are lovely creations, alive with colour. It is enjoyable to spend a bit of time marvelling at the structures within number that prime factorisation reveals, rather than simply noting their existence (see *figure* 6). It reminds me that maths is a subject to be understood and enjoyed for itself, not as a 'functional' means to an end.

Crossing paths with Eratosthenes' Sieve was an unhappy experience early in my mathematical journey, but I look upon it now with rather different eyes. The algorithm works by the recognition of the patterns in which prime factors occur, but when I first used it, I failed to take any time to 'stop and smell the roses' in those patterns. Perhaps my new version of a number grid that indulges those patterns will entertain or at least engage others to look upon primes in a new light.

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Notes

1 The teacher probably said 'whole number', though my memory is hazy, and 'divides' meant 'divides without remainder'.

2 You are probably aware that the 'Sieve of Eratosthenes' is the name given to the ancient Greek algorithm that finds prime numbers by exclusion of multiples. Starting with a list of integers 2, 3, 4,…, you circle the lowest number and cross out all the multiples of that number, using repeated addition to identify multiples. You repeat this process until all numbers are either crossed out or circled. The circled numbers are the primes.

3 In many ways, 30 is a better example, because it has three distinct prime factors: 30 = 2×3×5. However, students are likely to have considered 12 in their own investigations, and it is therefore probably a more relevant example to choose. Eight is similar to 12, but since 8 = 2×2×2, some students might mistakenly think that the reason for concentrating on 8 is that all its factors are the same.

4 Here is perhaps an opportunity to consider the role of brackets both in language and in mathematics?

5 The 'argument from authority' is familiar to all: I say that X (*sotto voce*: and *I* should know), so X.

A free piece of software by Alec for looking at these primitive visualisations is available from www.ptolemy.co.uk/primitives/ and www.atm.org.uk/mt207.

figure 6